

Five-dimensional metrics of Petrov type 22

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ABSTRACT: We classify all five-dimensional Einstein manifolds that are static, have an $SO(3)$ isometry group and have Petrov type 22. We use this classification to show that the localized black hole in the Randall–Sundrum scenario necessarily has Petrov type 4.

KEYWORDS: Classical Theories of Gravity, Black Holes, Extra Large Dimensions.

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1. Introduction

In this article, we study the metric of a black hole that is localized on a brane in the five-dimensional AdS-space. This metric is of interest because it presumably describes the gravitational collapse of matter trapped on a brane in the Randall–Sundrum scenario [1]. It was argued in ref. [2] that this black hole is described by a “black cigar” in five dimensions. However the analytical form of the black cigar metric is not known. For an attempt to find this metric using a restricted ansatz, see ref. [3]. Some numerical approximations can be found in refs. [4, 5]. The analytical form of the metric of a black hole living on a membrane in four-dimensional AdS-space was constructed in ref. [6].

We use the five-dimensional Petrov classification [7] to look for this metric. We show that this unknown metric should have Petrov type 22 or type 4. In the latter case, we do not have additional constraints on the Weyl tensor. Imposing Petrov type 22, however, leads to an additional constraint on the Weyl tensor. This makes it easier to solve Einstein’s equations. We therefore look for the metric of the localized black hole within the class of metrics having Petrov type 22. This method is similar to the method used by Kerr for constructing the

rotating black hole in four dimensions. This metric was found while sieving through the metrics of Petrov type D .

The outline of this article is as follows. In Section 2, we give a short review of the five-dimensional Petrov classification. There, we also show that the Petrov type of a black hole that is localized on a brane should have Petrov type 4 or 22. In Section 3, we classify all metrics that have Petrov type 22, are static¹ and have a space-like $SO(3)$ isometry group. In Section 4, we classify all metrics that have Petrov type 22, are static and have a space-like $SO(3)$ isometry group. From these classifications, it follows that the black hole we were looking for, does not have type 22. Hence, unfortunately, we did not find the analytical form of its metric.

2. Review of the five-dimensional Petrov classification

We only give a brief review of this classification, a longer discussion can be found in [7]. We need to introduce two objects, the *Weyl spinor* and the *Weyl polynomial*.

The Weyl spinor Ψ_{abcd} is the spinorial translation of the Weyl tensor C_{ijkl}

$$\Psi_{abcd} = (\gamma_{ij})_{ab}(\gamma_{kl})_{cd}C^{ijkl}.$$

Here, $\gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$, where γ_i are the γ -matrices in 5 dimensions. These are 4×4 matrices. It can be verified that the Weyl spinor is symmetric in all its indices.

The Weyl polynomial Ψ is a homogeneous polynomial of degree four in four variables:

$$\Psi = \Psi_{abcd}x^ax^bx^cx^d. \quad (2.1)$$

The Petrov type of a given Weyl tensor is nothing else than the number and multiplicity of the irreducible factors of Ψ . In this way, we get 12 different Petrov types, which are depicted in figure 1. We use the following notation. The number denotes the degree of the irreducible factors and underbars denote the multiplicities. For example, a Weyl polynomial which can be factorized into two different factors, each having degree 2, has Petrov type 22. As a second example, a Weyl polynomial which can be factorized in a polynomial of degree 2 times the square of a polynomial of degree 1 has Petrov type 211. The arrows between the different Petrov types denote increasing specialization of the Weyl tensor. In accordance to the literature on the four-dimensional Petrov classification, all Weyl tensors different from type 4 are called algebraically special.

Example: the black string in AdS

The metric of a black string in AdS reads [2]

$$ds^2 = \frac{l^2}{z^2} (U dt^2 + U^{-1} dr^2 + dz^2 + r^2 d\Omega_2^2), \quad (2.2)$$

¹although it was argued in ref. [8] that the localized black hole should be time-dependent. See ref. [9] for an AdS/CFT argument.

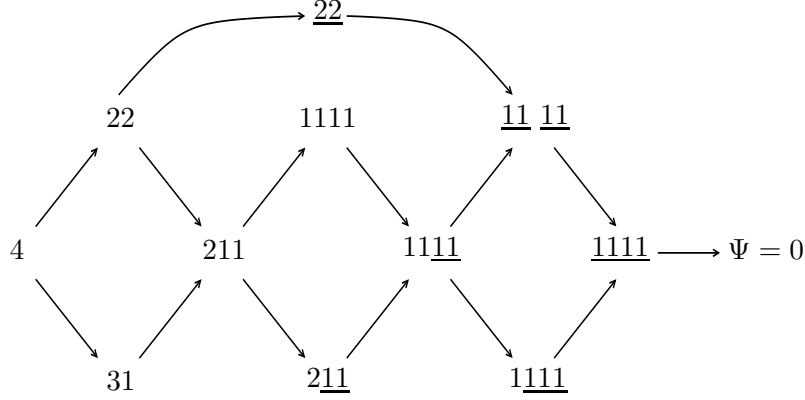


Figure 1: The 12 different Petrov types in 5 dimensions.

where $U(r) = 1 - \frac{2m}{r}$ and $d\Omega_2^2$ is the metric on the unit 2-sphere. If we choose the tetrad as

$$\begin{aligned} e_t &= \frac{z}{l} U^{-1/2} \partial_t & e_r &= \frac{z}{l} U^{1/2} \partial_r & e_z &= \frac{z}{l} \partial_z \\ e_\theta &= \frac{z}{lr} \partial_\theta & e_\phi &= \frac{z}{lr \sin \theta} \partial_\phi, \end{aligned}$$

we find after some algebra the Weyl polynomial

$$\Psi = -\frac{48mz^2}{l^2 r^3} (T^2 X^2 + X^2 Y^2 + T^2 Z^2 + Y^2 Z^2 - 4TXYZ).$$

The metric (2.2) has Petrov type 22 because Ψ can be factorized into two polynomials of degree 2

$$\Psi = -\frac{48mz^2}{l^2 r^3} (TX + iXY - iTZ - YZ) (TX - iXY + iTZ - YZ).$$

Far from the AdS-horizon, the localized black hole will look like this black string [2]. Because the black string has type 22, it follows from Figure 1 that the localized black hole should have Petrov type 22 or 4. In the next Section, we will look for the metric of the localized black hole by classifying the metrics of type 22.

3. Classification of metrics of type 22

In this section, we classify all metrics that are static, have an $SO(3)$ -isometry group and have Petrov type 22. These metrics satisfy Einstein's equations. Hence, the Ricci-tensor satisfies $R_{\mu\nu} = 4\Lambda g_{\mu\nu}$. We will assume in the rest of the article that $\Lambda \neq 0$. The classification in the case of $\Lambda = 0$ can be found in ref. [7].

A calculation shows that the most general ansatz for the Weyl spinor Ψ_{abcd} has 4 independent components.

$$\begin{aligned}
\Psi_{1114} &= \Psi_{1444} = \Psi_{2223} = \Psi_{2333} = 24i\varphi_4 \\
\Psi_{1123} &= \Psi_{1224} = \Psi_{1334} = \Psi_{2344} = -8i\varphi_4 \\
\Psi_{1111} &= \Psi_{2222} = \Psi_{3333} = \Psi_{4444} = -24(\varphi_2 + \varphi_3) \\
\Psi_{1133} &= \Psi_{2244} = 8(\varphi_2 + \varphi_3) \\
\Psi_{2233} &= \Psi_{1144} = 8(\varphi_2 - 3\varphi_3) \\
\Psi_{1122} &= \Psi_{3344} = 8(-2\varphi_1 - \varphi_2 + \varphi_3) \text{ and } \Psi_{1234} = 8(\varphi_1 + \varphi_3)
\end{aligned} \tag{3.1}$$

Here, the four functions φ_i depend on the coordinates r and z . The above linear combinations are chosen in such a way that the functions φ_i transform nicely under tetrad transformations (see appendix A).

A metric has Petrov type 22 if by definition the Weyl polynomial obtained from the Weyl spinor (3.1) can be factorized into two factors of degree 2. In [7], we have shown that this is the case if and only if one of the following four conditions is satisfied

$$\begin{aligned}
\text{Case A: } \varphi_3 = \varphi_4 = 0 & \quad \text{Case C: } \varphi_1^2 = \varphi_3^2 + \varphi_4^2 \\
\text{Case B: } \varphi_1 + \varphi_2 = 0 & \quad \text{Case D: } \varphi_2^2 = \varphi_3^2 + \varphi_4^2
\end{aligned} \tag{3.2}$$

The classification is then as follows.

The metric has type 22 if and only if it is one of the following metrics.

1. The black string in AdS [2]

$$ds^2 = \frac{l^2}{z^2} (U dt^2 + U^{-1} dr^2 + dz^2 + r^2 d\Omega^2), \tag{3.3}$$

where $U(r) = 1 - \frac{2m}{r}$ and $d\Omega^2$ is the metric on the unit 2-sphere.

2. A wrapped product built on a dS-Schwarzschild space

$$ds^2 = dz^2 + \sin^2(\sqrt{\Lambda} z) \left[\left(1 - \frac{2m}{r} - \Lambda r^2 \right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \Lambda r^2} + r^2 d\Omega^2 \right] \tag{3.4}$$

This metric is part of the so called RS-AdS-Schwarzschild black hole [15, 16]

3. A cosmological model in which space is the product of a two-sphere and a two-dimensional manifold of constant curvature. Hence, the metric reads

$$ds^2 = dz^2 + a(z)^2 d\Omega^2 + b(z)^2 dM^2, \tag{3.5}$$

where M can be \mathbb{E}^2 , S^2 or \mathbb{H}^2 . The functions $a(z)$ and $b(z)$ are determined by Einstein's equations. We do not know the analytical expression of the general solution. A particular solution is $a(z) = b(z) = \sin(\sqrt{\Lambda} z)/\sqrt{3\Lambda}$. See ref. [17] for a classification of all 4+1-dimensional cosmological models with spatial hypersurfaces that are homogeneous, connected and simply connected.

The derivation of this classification is tedious, it can be found in appendices D and E. We now prove² that the localized black hole on a brane has Petrov type 4. In Section 2, we argued that the metric of the localized black hole has type 22 or type 4. This metric does not appear in our classification of metrics of type 22. Hence, it has Petrov type 4.

4. Classification of metrics of type 22

A metric has Petrov type 22 if by definition the Weyl polynomial is the square of a polynomial of degree 2. This is the case if and only if one of the following three conditions is satisfied

$$\begin{aligned} \text{Case AC: } & \varphi_1 = \varphi_3 = \varphi_4 = 0 \\ \text{Case AD: } & \varphi_2 = \varphi_3 = \varphi_4 = 0 \\ \text{Case BC: } & \varphi_1 + \varphi_2 = 0 \quad \text{and} \quad \varphi_1^2 = \varphi_3^2 + \varphi_4^2 \end{aligned} \tag{4.1}$$

These conditions are combinations of the conditions (3.2). This is reflected in our notation. The classification is then as follows.

The metric has type 22 if and only if it is one of the following metrics.

1. The product of the spheres S^3 and S^2

$$ds^2 = \frac{1}{2\Lambda} d\Omega_3^2 + \frac{1}{4\Lambda} d\Omega_2^2 \tag{4.2}$$

2. The generalized AdS-Schwarzschild metric [14]

$$ds^2 = U(r) dt^2 + U^{-1}(r) dr^2 + r^2 dM^2, \tag{4.3}$$

where $U(r) = k - \frac{m}{r^2} + \Lambda r^2$ and M is a space of constant curvature. If $M = S^3$, \mathbb{E}^3 or \mathbb{H}^3 , then $k = 1, 0, -1$ respectively. If M is the unit three-sphere, then the metric reduces to AdS-Schwarzschild.

Details of the classification can be found in appendix C.

5. Conclusions

In this article, we have given a classification of five-dimensional Einstein manifolds that are static, have an $SO(3)$ isometry group and have Petrov type 22. We have used this classification to prove that the localized black hole on a brane is not algebraically special. Some topics of further research are the following.

²We remind the reader that we have assumed that the metric of the localized black hole is static. If the results in ref. [8] and ref. [9] are valid, this restriction is too strong and our proof breaks down.

1. One can further refine the five-dimensional Petrov classification as given in [7]. Indeed, it should be possible to subdivide the Petrov types for which the Weyl polynomial can not be completely factorized into smaller subtypes (i.e. the Petrov types 4, 31, 22, 22 and 211). This project is essentially the determination of an invariant classification of the polynomials of degree four in four variables, hence belongs to 19th century's "invariant theory". See ref. [19] for an introduction to classical invariant theory.
2. Classify all Einstein metrics of type 22 without assuming additional isometries. This classification is a higher-dimensional generalization of Kinnersley's classification of four-dimensional metrics of Petrov type D . This could lead to the discovery of new five-dimensional metrics. It would shed some light on the possible solutions of the five-dimensional Einstein's equations. One knows that the set of black holes is much richer in five dimensions than in four dimensions. See e.g. ref. [18] for a five-dimensional rotating black string.

Before attacking the general classification, it is perhaps better in view of refs. [8] and [9] to start by classifying time-dependent metrics with an $SO(3)$ isometry group.

3. As far as I know, almost nothing is known about the five-dimensional Petrov classification. It would be good to make a thorough study of it.

Acknowledgments

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A. Tetrad transformation on the Weyl spinor

Under tetrad rotations ($s = +1$) and reflections ($s = -1$)

$$\begin{pmatrix} e_r \\ e_z \end{pmatrix} \rightarrow \begin{pmatrix} \cos \chi & \sin \chi \\ -s \sin \chi & s \cos \chi \end{pmatrix} \begin{pmatrix} e_r \\ e_z \end{pmatrix},$$

the Weyl spinor transforms as

$$\varphi_1 \rightarrow \varphi_1, \quad \varphi_2 \rightarrow \varphi_2, \quad \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} \rightarrow \begin{pmatrix} \cos 2\chi & -\sin 2\chi \\ s \sin 2\chi & s \cos 2\chi \end{pmatrix} \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}.$$

From this, we have the important result that we can always do a tetrad rotation to put $\varphi_4 = 0$. On top of this, we can specify the sign of φ_3 by an additional reflection.

B. The field equations

The vacuum Einstein equations (notice our non-standard normalization of the cosmological constant)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 14\Lambda g_{\mu\nu} = 0 \tag{B.1}$$

are equivalent to the system

$$d\omega + \omega \wedge \omega = \Omega \quad (\text{B.2})$$

$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0, \quad (\text{B.3})$$

where the Riemann tensor Ω is expressed in terms of the Weyl spinor Ψ_{abcd} (3.1) and the Ricci tensor, which is proportional to the metric by (B.1). The connection ω is the Levi-Civita connection associated with the tetrad. The most general ansatz for the tetrad of a static space-time with $SO(3)$ symmetry reads

$$\begin{aligned} e_t &= A \partial_t \\ e_r \text{ and } e_z &: \text{linear combinations of } \partial_r \text{ and } \partial_z \\ e_\theta &= \sigma \partial_\theta \\ e_\phi &= \frac{\sigma}{\sin \theta} \partial_\phi \end{aligned} \quad (\text{B.4})$$

In the above, A and σ are functions of the coordinates r and z . The non-zero commutators are

$$[e_t, e_r] = \mu e_t, \quad [e_t, e_z] = \nu e_t, \quad (\text{B.5a})$$

$$[e_r, e_\theta] = \kappa e_\theta, \quad [e_z, e_\theta] = \lambda e_\theta, \quad (\text{B.5b})$$

$$[e_r, e_\phi] = \kappa e_\phi, \quad [e_z, e_\phi] = \lambda e_\phi, \quad (\text{B.5c})$$

$$[e_r, e_z] = \delta e_r + \varepsilon e_z \quad (\text{B.5d})$$

$$[e_\theta, e_\phi] = -\cot \theta \sigma e_\phi, \quad (\text{B.5e})$$

where $\mu, \nu, \delta, \varepsilon, \kappa, \lambda$ and σ are functions of the coordinates r and z . We will split the resulting field equations into two blocks.

The equations from (B.2)

The definition of the curvature leads to two algebraic equations

$$\sigma^2 - \kappa^2 - \lambda^2 - \Lambda - 3\varphi_1 - \varphi_2 = 0 \quad (\text{B.6})$$

$$\kappa\mu + \lambda\nu - \Lambda + \varphi_1 - \varphi_2 = 0 \quad (\text{B.7})$$

and a set of first order differential equations. The ones involving the vector e_r are

$$e_r(\sigma) = \sigma\kappa \quad (\text{B.8a})$$

$$e_r(\varepsilon) - e_z(\delta) = \delta^2 + \varepsilon^2 + \varphi_1 + 3\varphi_2 + \Lambda \quad (\text{B.8b})$$

$$e_r(\kappa) = \kappa^2 - \delta\lambda - \varphi_1 - \varphi_2 - 2\varphi_3 + \Lambda \quad (\text{B.8c})$$

$$e_r(\lambda) = \kappa\delta + \kappa\lambda + 2\varphi_4 \quad (\text{B.8d})$$

$$e_r(\mu) = -\nu\delta - \mu^2 - \varphi_1 + \varphi_2 - 4\varphi_3 - \Lambda \quad (\text{B.8e})$$

$$e_r(\nu) = -\nu\mu + \delta\mu + 4\varphi_4 \quad (\text{B.8f})$$

and the ones involving the vector e_z are

$$e_z(\sigma) = \sigma\lambda \quad (\text{B.9a})$$

$$e_z(\kappa) = -\varepsilon\lambda + \kappa\lambda + 2\varphi_4 \quad (\text{B.9b})$$

$$e_z(\lambda) = \varepsilon\kappa + \lambda^2 - \varphi_1 - \varphi_2 + 2\varphi_3 + \Lambda \quad (\text{B.9c})$$

$$e_z(\mu) = -\nu\varepsilon - \nu\mu + 4\varphi_4 \quad (\text{B.9d})$$

$$e_z(\nu) = -\nu^2 + \varepsilon\mu - \varphi_1 + \varphi_2 + 4\varphi_3 - \Lambda \quad (\text{B.9e})$$

The equations from (B.3)

The Bianchi identity leads the following set of equations

$$\begin{aligned} e_r(\varphi_1) &= \frac{5}{2}\kappa\varphi_1 + \frac{1}{2}(\kappa + \mu)\varphi_2 + \frac{1}{2}(4\kappa + \mu)\varphi_3 - \frac{1}{2}(\nu + 4\lambda)\varphi_4 \\ e_r(\varphi_2) &= \frac{1}{2}\kappa\varphi_1 + \frac{1}{2}(5\kappa - 3\mu)\varphi_2 - \frac{1}{2}(4\kappa + 3\mu)\varphi_3 + \frac{1}{2}(3\nu + 4\lambda)\varphi_4 \\ e_r(\varphi_3) &= e_z(\varphi_4) + \frac{1}{2}\kappa\varphi_1 - \frac{1}{2}(\kappa + \mu)\varphi_2 + (2\varepsilon + \kappa - \frac{1}{2}\mu)\varphi_3 \\ &\quad + (\frac{1}{2}\nu + 2\delta - \lambda)\varphi_4 \\ e_r(\varphi_4) &= -e_z(\varphi_3) - \frac{1}{2}\lambda\varphi_1 + \frac{1}{2}(\nu + \lambda)\varphi_2 + (-\frac{1}{2}\nu - 2\delta + \lambda)\varphi_3 \\ &\quad + (2\varepsilon + \kappa - \frac{1}{2}\mu)\varphi_4 \\ e_z(\varphi_1) &= \frac{5}{2}\lambda\varphi_1 + \frac{1}{2}(\nu + \lambda)\varphi_2 - \frac{1}{2}(\nu + 4\lambda)\varphi_3 - \frac{1}{2}(4\kappa + \mu)\varphi_4 \\ e_z(\varphi_2) &= \frac{1}{2}\lambda\varphi_1 + \frac{1}{2}(-3\nu + 5\lambda)\varphi_2 + \frac{1}{2}(3\nu + 4\lambda)\varphi_3 + \frac{1}{2}(4\kappa + 3\mu)\varphi_4 \end{aligned} \quad (\text{B.10})$$

C. Type 22: solutions in cases AC, AD and BC

In this appendix, we classify all solutions of the field equations given in appendix B that are of Petrov type 22. We point out that if all components of the Weyl spinor (3.1) vanish, the space has constant curvature. Indeed, a conformally flat metric which is Einstein, is necessarily of constant curvature.

Case AC: $\varphi_1 = \varphi_3 = \varphi_4 = 0$

From the Bianchi equations (B.10), we immediately get $(\kappa + \mu)\varphi_2 = 0$ and $(\lambda + \nu)\varphi_2 = 0$. Hence, we have two possibilities. The first one is $\varphi_2 = 0$. Then all components of the Weyl spinor are zero and we obtain a space of constant curvature. The second possibility is $\kappa + \mu = 0$ and $\lambda + \nu = 0$. Then we obtain from (B.6) and (B.7) $\sigma = 0$. This is not a good solution of the field equations, because in this case the tetrad (B.4) is degenerate. Therefore, in case AC, we obtain only the five-sphere.

Case AD: $\varphi_2 = \varphi_3 = \varphi_4 = 0$

This case is similar to the previous one. From the Bianchi equations (B.10), we get $\kappa\varphi_1 = 0$ and $\lambda\varphi_1 = 0$. If $\varphi_1 = 0$, we have a sphere. If $\kappa = \lambda = 0$, we find from equations (B.7) and (B.6) $\sigma^2 = 4\Lambda$. Hence, the metric reads

$$ds^2 = dM_3^2 + \frac{1}{4\Lambda}d\Omega_2^2,$$

where Ω_2 is the unit two-sphere. This metric is Einstein if and only if M_3 is an Einstein space. As is well-known — see for example [13, Ex. 28.2] — a three-dimensional Einstein space is necessarily of constant curvature. Therefore M_3 is a sphere. All in all, Case AD leads to the five-sphere or to the product of two spheres (4.2).

Case BC: $\varphi_1 + \varphi_2 = 0$ and $\varphi_1^2 = \varphi_3^2 + \varphi_4^2$

As shown in appendix A, we can choose our tetrad in such a way that $\varphi_4 = 0$ and $\varphi_3 = \varphi_1$. We will assume that $\varphi_1 \neq 0$ otherwise, we obtain the five-sphere. From the Bianchi equations (B.10), we obtain $\nu\varphi_1 = \delta\varphi_1 = (\kappa - \varepsilon)\varphi_1 = 0$. Hence, we have $\nu = \delta = \kappa - \varepsilon = 0$. From equation (B.7), we get $\varphi_1 = -1/2 \kappa\mu + 1/2 \Lambda$. Inserting all this into equation (B.8) and (B.9) leads to the following equations

$$e_r(\kappa) = \kappa^2 + \kappa\mu \quad (\text{C.1a}) \quad e_z(\kappa) = 0 \quad (\text{C.1d})$$

$$e_r(\lambda) = \kappa\lambda \quad (\text{C.1b}) \quad e_z(\lambda) = \kappa^2 + \lambda^2 - \kappa\mu + 2\Lambda \quad (\text{C.1e})$$

$$e_r(\mu) = 3\kappa\mu - \mu^2 - 4\Lambda \quad (\text{C.1c}) \quad e_z(\mu) = 0 \quad (\text{C.1f})$$

We will resolve two cases $\kappa \neq 0$ and $\kappa = 0$.

- Case BC.1: $\kappa \neq 0$

From the commutation relation (B.5d), we see that we can always find a coordinate system in which $e_r = -r\kappa\partial_r$ and $e_z = 1/r\partial_z$. The solution of the differential equations (C.1) then reads

$$\kappa = \left(\frac{C_1}{r^4} + \frac{C_2}{r^2} - \Lambda \right)^{1/2}, \quad \lambda = -\frac{\sqrt{C_2}}{r} \cot(\sqrt{C_2}z),$$

and

$$\mu = \left(\frac{C_1}{r^4} + \Lambda \right) \left(\frac{C_1}{r^4} + \frac{C_2}{r^2} - \Lambda \right)^{-1/2}.$$

Here C_1 and C_2 are two arbitrary integration constants. We have used a shift on the coordinate z to eliminate the third integration constant. Finally, the remaining tetrad vectors e_t and e_θ are determined from the commutation relation (B.5a) and the algebraic equation (B.6) respectively

$$e_t = \left(C_2 + \frac{C_1}{r^2} - \Lambda r^2 \right)^{-1/2} \partial_t \quad \text{and} \quad \sigma^2 = \frac{C_2}{r^2 \sin^2(\sqrt{C_2}z)}.$$

This solution is the metric (4.3).

- Case BC.2: $\kappa = 0$

Because $\delta = \varepsilon = 0$, we can always find a coordinate system in which $e_r = \partial_r$ and $e_z = \partial_z$. It then follows from (B.6) and (B.5a) that the metric is of the form

$$ds^2 = W(r)dt^2 + dr^2 + dz^2 + \frac{1}{\sigma(r)^2}d\Omega^2.$$

Hence, it is the product of two Einstein spaces, leading to the product of two spheres (4.2).

D. Type 22: solutions in cases A, B and C

In this appendix, we classify all solutions of the field equations given in appendix B, subject to the conditions of case A, B and C of (3.2).

Case A: $\varphi_3 = \varphi_4 = 0$

The Bianchi identities give

$$\kappa\varphi_1 - (\kappa + \mu)\varphi_2 = 0 \quad \text{and} \quad -\lambda\varphi_1 + (\lambda + \nu)\varphi_2 = 0. \quad (\text{D.1})$$

Because we are not interested in the trivial case $\varphi_1 = \varphi_2 = 0$, it follows from these equations that $\kappa\nu = \lambda\mu$. We choose our tetrad such that $\nu = 0$. At this point, it is necessary to resolve two cases: $\mu = 0$ and $\mu \neq 0$.

- Case A.1: $\mu = 0$

From equations (B.7) and (B.8e), we have $\Lambda = 0$. We are not interested in this case, a discussion can be found in [7].

- Case A.2: $\mu \neq 0$

In this case, we have $\lambda = 0$. From equation (B.8f) we find $\delta = 0$, and from equations (B.7), (B.9e), (B.9c) and (D.1) we obtain

$$\kappa(\kappa + \mu) = 0, \quad \varphi_1 = \frac{\varepsilon}{2}(\kappa + \mu) \quad \text{and} \quad \varphi_2 = \frac{\kappa}{2}(\varepsilon + \mu).$$

Hence we have $\varphi_1 = 0$ or $\varphi_2 = 0$, leading to cases AC or AD. The discussion in appendix C shows that this leads to the five-sphere or the product of two spheres (4.2).

Case B: $\varphi_1 + \varphi_2 = 0$

We choose the tetrad in such a way that $\varphi_4 = 0$. From the Bianchi equations (B.10), we get $\mu(\varphi_1 - \varphi_3) = 0$ and $\nu(\varphi_1 + \varphi_3) = 0$. Hence, there are two possibilities. The first one is $\mu \neq 0$ or $\nu \neq 0$. Then we have $\varphi_1 = \varphi_3$ or $\varphi_1 = -\varphi_3$. Therefore the metric has type 22. This case is treated in case BC of appendix C. The second possibility is $\mu = \nu = 0$. Then, it follows from (B.5a) that the manifold is a product manifold with metric of the form $ds^2 = dt^2 + dM^2$. Λ has to be zero for this metric to be Einstein.

Case C: $\varphi_1^2 = \varphi_3^2 + \varphi_4^2$

We choose our tetrad such that $\varphi_4 = 0$ and $\varphi_3 = \varphi_1$. A similar analysis as the one in [7] shows that we have either case AC, BC or

$$\lambda = 0, \quad \nu = 0, \quad \varphi_1 = -\frac{1}{8}(\mu + \varepsilon)(\kappa + \mu) \quad \text{and} \quad \varphi_2 = \frac{1}{8}(\mu + \varepsilon)(3\kappa - 2\varepsilon + \mu).$$

Inserting these relations into (B.9) leads to $\Lambda = 0$.

E. Type 22: solutions in case D, $\varphi_2^2 = \varphi_3^2 + \varphi_4^2$

We choose the tetrad in such a way that $\varphi_4 = 0$ and $\varphi_3 = -\varphi_2$. From the Bianchi identities, it follows that $\varphi_2 = 0$ or that $\delta = \nu$. In the former case, the discussion in case AD shows that this leads to the five-sphere or the product of two spheres. We will now proceed with the latter case $\delta = \nu$. An analysis similar to the one in [7] shows that the problem is split into three cases

$$\begin{aligned} \text{D.1} \quad & \varepsilon = \kappa \neq 0 \quad \text{and} \quad \varphi_1 + \varphi_2 = 0, \\ \text{D.2} \quad & \varepsilon = 0, \quad \kappa \neq 0 \quad \text{and} \quad \varphi_1 + 3\varphi_2 = 0, \\ \text{D.3} \quad & \varepsilon = \kappa = 0. \end{aligned}$$

Case D.1: $\varepsilon = \kappa \neq 0$ and $\varphi_1 + \varphi_2 = 0$

This case is treated in case BC of appendix C.

Case D.2: $\varepsilon = 0$, $\kappa \neq 0$ and $\varphi_1 + 3\varphi_2 = 0$

Inserting the relation $\varphi_1 + 3\varphi_2 = 0$ into the Bianchi equations gives $\lambda = -\nu$. From equation (B.7), we get $\varphi_2 = 1/4(\kappa\mu - \nu^2 - \Lambda)$. Inserting these results into equations (B.8) and (B.9) gives

$$e_r(\mu) = -3\nu^2 - \mu^2 + 2\kappa\mu - 3\Lambda \quad (\text{E.1a}) \quad e_z(\mu) = -\nu\mu \quad (\text{E.2a})$$

$$e_r(\nu) = 0 \quad (\text{E.1b}) \quad e_z(\nu) = -\nu^2 - \Lambda \quad (\text{E.2b})$$

$$e_r(\kappa) = \kappa^2 + \kappa\mu \quad (\text{E.1c}) \quad e_z(\kappa) = -\nu\kappa \quad (\text{E.2c})$$

We choose coordinates in such a way that $e_r = A\partial_r$ and $e_z = B\partial_z$. From the commutation relation (B.5d), it follows that B depends only on the coordinate z , hence, by a coordinate transformation, we can put $B = \sqrt{\Lambda}$. The solution of (E.2b) splits into two cases.

- ν is not constant

Then we have $\nu = \sqrt{\Lambda} \cot z$. We have removed the integration constant by a coordinate transformation on z . From equations (E.2), we have

$$\mu = \frac{f(r)}{\sin z} \quad \text{and} \quad \kappa = \frac{g(r)}{\sin z}.$$

From the commutation relations, it follows that we can take $e_r = -\frac{rg(r)}{\sin z}\partial_r$ without loosing generality. The solution of (E.1) then reads

$$f(r)g(r) = \Lambda - \frac{m}{r^3} \quad \text{and} \quad g(r) = \frac{1}{r} \left[C_1 - \frac{2m}{r} - \Lambda r^2 \right]^{1/2}.$$

This leads to the metric (3.4).

- ν is constant

In a similar way as above, one can show that this case leads to the black string in AdS (2.2).

Case D.3: $\varepsilon = \kappa = 0$

From (B.7), we get $\varphi_1 = \varphi_2 - \lambda\nu + \Lambda$. From (B.8) and (B.9) we obtain the following equations

$$\begin{aligned} e_r(\mu) &= -\nu^2 - \mu^2 + \lambda\nu + 4\varphi_2 - 2\Lambda & e_z(\mu) &= -\nu\mu \\ e_r(\nu) &= 0 & e_z(\nu) &= -\nu^2 + \lambda\nu - 4\varphi_2 - 2\Lambda \\ e_r(\lambda) &= 0 & e_z(\lambda) &= \lambda^2 + \lambda\nu - 4\varphi_2 \end{aligned} \quad (\text{E.3}) \quad (\text{E.4})$$

We choose coordinates in such a way that $e_r = A\partial_r$ and $e_z = B\partial_z$. From the commutation relation (B.5d), it follows that we can do a coordinate transformation to make A and B both depend only on z . The solution splits into three cases

- $\mu = 0$

From (B.6) and (B.5a), it follows that the metric can be written as

$$ds^2 = \frac{1}{A(z)^2}(dt^2 + dr^2) + \frac{1}{B(z)^2}dz^2 + \frac{1}{\sigma(z)^2}d\Omega^2.$$

This metric belongs to the class (3.5), where the factor is a two-plane. We have not been able to find an analytical form of the general solution of (E.3) and (E.4).

- $\mu \neq 0$ and $e_r(\mu) = 0$

In this cases, the metric can be written as

$$ds^2 = \frac{1}{A(z)^2}(dr^2 + e^{-2Cr}dt^2) + \frac{1}{B(z)^2}dz^2 + \frac{1}{\sigma(z)^2}d\Omega^2.$$

This metric belongs to the class (3.5), where the second factor is the two-sphere if $C^2 < 0$ and the two-dimensional hyperbolic space if $C^2 > 0$. We have not been able to find an analytical form of the general solution of (E.3) and (E.4).

- $\mu \neq 0$ and $e_r(\mu) \neq 0$

This case leads to the same metrics as above.

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